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A Study of the Optimum Transversal Contour of a Body at Hypersonic Speeds Using Direct Methods

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by

ANGELO MIELE(*)

SUMMARY

This paper refers to a slender conical body whose length and base area are given and considers the problem of determining the transversal contour which minimizes the overall drag (sum of the pressure drag and the friction drag) under the assumption that the pressure coefficient satisfies Newton's impact law and that the friction coefficient is constant. Direct methods are employed, and the analysis is confined to a body whose cross section is either a regular polygon or is composed of a basic circle external to which are symmetric segments of a logarithmic spiral. The analysis shows that the optimum solution is governed by the friction parameter K_f, a parameter which is proportional to the ratio of the friction coefficient to the cube of the average thickness ratio of the body.

For the class of pyramidal bodies, the optimum transversal contour is a triangle for $K_f \lesssim 0.47$ and a circle for $K_f > 1$. For intermediate values of the friction parameter, the optimum number of sides n satisfies the inequality $3 < n < \infty$ and is such that the friction drag is twice the pressure drag.

(*) Director of Astrodynamics and Flight Mechanics

For a body whose cross section is a circle with n symmetric arcs of a logarithmic spiral superimposed, the optimum number of segments n depends not only on the friction parameter but also on the ratio of the highest to the lowest radius of the cross section. If the value of this ratio is two, the optimum number of segments satisfies the inequality $2 < n < \infty$ for $K_f \lesssim 0.9$ and is such that the friction drag is twice the pressure drag; it decreases as the friction parameter increases and becomes $n = 2$ for $K_f \cong 0.9$. For any higher value of the friction parameter, the optimum number of segments remains equal to two.

Finally, the simultaneous optimization of the longitudinal and transversal contours is considered in the appendix in connection with a power body with a polygonal cross section. While the results relative to the transversal contour are identical with those obtained for conical bodies, the longitudinal contour changes to a considerable degree depending on the friction parameter. This contour follows the 3/4-power law for $K_f = 0$, is convex for $0 < K_f \lesssim 0.47$, is a straight line for $0.47 \lesssim K_f < 1$, and is concave for $K_f > 1$.

1. INTRODUCTION

In a recent paper (Ref. 1), Chernyi and Gonor investigated the simultaneous determination of the longitudinal and transversal contours of the three-dimensional body minimizing the drag at hypersonic speeds for the case where the length is prescribed, the base area is prescribed, and its contour is required to lie outside a circle of radius r_o. They restricted the analysis to the class of bodies which are slender in the longitudinal sense and homothetic in the transversal sense, that is, such that each section perpendicular to the undisturbed flow direction is geometrically similar to any other. They employed two basic hypotheses: (a) the friction drag is neglected; and (b) the distribution of pressure coefficients is governed by Newton's impact law. Because of these hypotheses, they were able to separate the problem of the optimum longitudinal contour from that of the optimum transversal contour, to show that the optimum longitudinal contour follows a 3/4-power law, as well as to show that the optimum transversal contour has a starlike configuration. These authors also found that the drag coefficient decreases monotonically as the number of segments n composing the star increases; that, if n is in the order of 10, the drag coefficient is in the order of one-tenth or less of the drag coefficient of the equivalent body of revolution; and that, if the limiting process $n \to \infty$ is carried out, the drag coefficient of the optimum body tends to zero.

Owing to the nature of these results, a reexamination of the basic hypotheses is in order, that is, (a) neglecting the friction drag and (b) employing Newton's law in determining the pressure coefficients. While the discussion of the validity of Newton's impact law is left to a forthcoming paper, the following sections are concerned with the effect of the friction drag on the optimum transversal contour. To do this, Newton's impact law is retained for the distribution of pressure coefficients while the friction coefficient is regarded to be constant everywhere. Furthermore, direct methods are employed, and the analysis is confined to the class of bodies (a) which are conical in the longitudinal sense and (b) which, in the transversal sense, are either polygonal or composed of a basic circle with symmetric arcs of a logarithmic spiral superimposed.

2. DETERMINATION OF THE DRAG

Consider two systems of coordinates: a Cartesian coordinate system x, y, z and a cylindrical coordinate system ρ, θ, z. With regard to the Cartesian coordinate system, the z-axis is identical with the undisturbed flow direction, and the xy-plane is perpendicular to the z-axis. For the cylindrical coordinate system, ρ is the distance of any point from the z-axis, and θ measures the angular position of this point with respect to the xz-plane (Fig. 1). Next, restrict the analysis to the class of bodies which are homothetic in the transversal sense, that is, those bodies whose shapes can be expressed in the form

$$\phi(\rho, \theta, z) \equiv \rho - r(\theta)\, f(z) = 0 \tag{1}$$

where $f(z)$ is a function describing the longitudinal contour and $r(\theta)$ is a function describing the transversal contour. Clearly, each cross section is geometrically similar to the base contour. If the length of the body is denoted by ℓ and if the coordinates ρ and z are measured in terms of ℓ, the base contour is described by $z = 1$. Furthermore, if the function f is chosen in such a way that $f(1) = 1$, then the equation $r(\theta)$ represents the transversal contour of the base in polar coordinates.

Now, denote by $\bar{u}_\rho$, $\bar{u}_\theta$, $\bar{u}_z$ the unit vectors of the cylindrical coordinate system, and observe that the direction of $\bar{u}_z$ is the same as the undisturbed flow direction. Also, denote by $\bar{n}$ the unit vector normal to the infinitesimal element of dimensionless wetted area $d\sigma$, and assume that $\bar{n}$ is positively oriented outward. Consequently, the aerodynamic drag per unit square length and dynamic pressure is given by

$$K_D = \frac{D}{q\ell^2} = \iint_\sigma \left[- C_p \bar{n} \cdot \bar{u}_z + C_f |\bar{n} \times \bar{u}_z|\right] d\sigma \tag{2}$$

where

$$C_p = 2(\bar{n} \cdot \bar{u}_z)^2 \tag{3}$$

is the pressure coefficient associated with Newton's impact law and C_f is the friction coefficient (assumed constant everywhere).

Since the unit vector $\bar{n}$ is perpendicular to the surface of the body, its direction is identical with that of the gradient of the function ψ. Hence, the relationship

$$\bar{n} = \frac{\nabla\psi}{|\nabla\psi|} \tag{4}$$

holds. After it is observed that

$$\nabla\psi = \frac{\partial\psi}{\partial\rho} \bar{u}_\rho + \frac{1}{\rho} \frac{\partial\psi}{\partial\theta} \bar{u}_\theta + \frac{\partial\psi}{\partial z} \bar{u}_z \tag{5}$$

and after Eq. (1) is accounted for, one deduces that

$$\bar{n} = \frac{\bar{u}_\rho - (r'/r)\bar{u}_\theta - r\dot{f}\bar{u}_z}{\sqrt{1 + (r'/r)^2 + (r\dot{f})^2}} \tag{6}$$

where

$$r' = \frac{dr}{d\theta} \quad , \quad \dot{f} = \frac{df}{dz} \tag{7}$$

After the infinitesimal element of wetted area is written in the form

$$d\sigma = \frac{rf}{\bar{n} \cdot \bar{u}_\rho} \, d\theta dz \tag{8}$$

and after Eqs. (2), (3), (6), and (8) are combined, laborious manipulations lead to the expression

$$K_D = \int_0^1 \int_0^{2\pi} \left[\frac{2r^4 f\dot{f}^3}{1 + (\dot{f}/r)^2 + (r\dot{f})^2} + C_f r f \sqrt{1 + (\dot{f}/r)^2} \right] dz d\theta \tag{9}$$

which reduces to

$$K_D = \int_0^1 \int_0^{2\pi} \left[\frac{2r^6 f\dot{f}^3}{r^2 + \dot{r}^2} + C_f f \sqrt{r^2 + \dot{r}^2} \right] dz d\theta \tag{10}$$

if the slender body approximation

$$(r\dot{f})^2 \ll 1 \tag{11}$$

is employed.

Consider, now, the class of bodies which are conical in the longitudinal sense and, for that reason, are represented by

$$f(z) = z \tag{12}$$

Since $\dot{f} = 1$, Eq. (10) simplifies to

$$K_D = K_{Dp} + K_{Df} \tag{13}$$

where

$$\begin{aligned} K_{Dp} &= \int_0^{2\pi} \frac{r^6}{r^2 + \dot{r}^2}\, d\theta \\ K_{Df} &= \frac{C_f}{2}\int_0^{2\pi} \sqrt{r^2 + \dot{r}^2}\, d\theta \end{aligned} \tag{14}$$

Several particular cases are now considered. For all these cases, the drag (13) will be calculated in combination with the dimensionless base area

$$S = \frac{1}{2}\int_0^{2\pi} r^2 d\theta \tag{15}$$

3. CIRCULAR CONTOUR

For a circular cone, the base contour is described by

$$r = \text{Const} \tag{16}$$

which implies that $\dot{r} = 0$. Since the dimensionless drag integrals and base area are given by

$$\begin{aligned} K_{Dp} &= 2\pi r^4 \\ K_{Df} &= C_f \pi r \\ S &= \pi r^2 \end{aligned} \tag{17}$$

the drag per unit square length and dynamic pressure becomes

$$K_D = \pi r(2r^3 + C_f) \tag{18}$$

and can be rewritten in the form

$$K_D = \frac{2S^2}{\pi} (1 + 2K_f) \tag{19}$$

where

$$K_f = \frac{C_f}{4} \left(\frac{\pi}{S}\right)^{3/2} \tag{20}$$

denotes the friction parameter. This parameter is proportional to the ratio of the friction coefficient to the cube of the thickness ratio and, hence, is a measure of the relative importance of the friction drag with respect to the pressure drag.

4. POLYGONAL CONTOUR

If the transversal contour of the body is a regular polygon with n sides, the equation of each side can be written in the form

$$r \cos\theta = r_o \tag{21}$$

where r_o denotes the minimum radius and θ an angular coordinate which is measured from the r_o-direction and is valid in the interval

$$-\frac{\pi}{n} \leq \theta \leq +\frac{\pi}{n} \tag{22}$$

After the definition

$$I = \int_{-\pi/n}^{+\pi/n} \frac{d\theta}{\cos^2\theta} = 2 \tan\frac{\pi}{n} \tag{23}$$

is introduced and it is observed that

$$\begin{aligned} K_{Dp} &= nr_o^4 I = 2nr_o^4 \tan\frac{\pi}{n} \\ K_{Df} &= \frac{nC_f r_o}{2} I = nC_f r_o \tan\frac{\pi}{n} \\ S &= \frac{nr_o^2}{2} I = nr_o^2 \tan\frac{\pi}{n} \end{aligned} \tag{24}$$

the drag per unit square length and dynamic pressure becomes

$$K_D = nr_o(2r_o^3 + C_f) \tan\frac{\pi}{n} \tag{25}$$

and can be rewritten in the form

$$K_D = \frac{2S^2}{\pi}\left(\frac{1}{x} + 2K_f\sqrt{x}\right) \tag{26}$$

In this equation, the friction parameter K_f is defined by Eq. (20), and the auxiliary variable x is defined as

$$x = \frac{n}{\pi}\tan\frac{\pi}{n} \tag{27}$$

and, therefore, is monotonically related to the number of sides in the interval of interest $3 \leq n < \infty$. This means that the variable x satisfies the inequality(*)

$$1 \leq x \leq \frac{3\sqrt{3}}{\pi} \tag{28}$$

with the lower bound corresponding to $n = \infty$ and the upper bound corresponding to $n = 3$.

(*)The limitation $n \geq 3$ is imposed in order to insure that the length of the sides of the polygon be finite. Theoretically speaking, however, one can conceive the possibility that $n = 2$; obviously, the area enclosed by this special polygon would be finite only if each side were infinitely long.

It is of interest to evaluate the ratio of the drag of the pyramidal body to that of the equivalent body of revolution (subscript R), that is, the body of revolution of equal length and base area. This ratio is given by

$$\frac{D}{D_R} = \frac{1 + 2K_f x\sqrt{x}}{x(1 + 2K_f)} \tag{29}$$

and, for a given friction parameter, varies with the auxiliary variable x and, hence, with the number of sides of the cross section. It is inferred from Eq. (29) that there exists a particular value of x (and, hence, a particular value of n) which minimizes the drag ratio. This particular value is defined by

$$x = \frac{1}{K_f^{2/3}} \tag{30}$$

and implies the following: <u>the optimum number of sides is such that the friction drag is twice the pressure drag</u>. This optimum number of sides can then be obtained from the expression

$$K_f = \left[\frac{\pi}{n} \cot \frac{\pi}{n}\right]^{3/2} \tag{31}$$

and the associated drag ratio is

$$\frac{D}{D_R} = \frac{3K_f^{2/3}}{1 + 2K_f} \tag{32}$$

providing that

$$\frac{\pi\sqrt{\pi}}{9\sqrt[4]{3}} \le K_f \le 1 \tag{33}$$

Should the friction parameter not satisfy the inequality (33), then the optimum solutions would be as follows:

$$K_f \le \frac{\pi\sqrt{\pi}}{9\sqrt[4]{3}} \qquad \left\{ \begin{aligned} x &= \frac{3\sqrt{3}}{\pi} \\ n &= 3 \\ \frac{D}{D_R} &= \frac{\pi\sqrt{\pi} + 18\sqrt[4]{3}\,K_f}{3\sqrt{3\pi}\,(1 + 2K_f)} \end{aligned} \right. \tag{34}$$

$$K_f \ge 1 \qquad \left\{ \begin{aligned} x &= 1 \\ n &= \infty \\ \frac{D}{D_R} &= 1 \end{aligned} \right. \tag{35}$$

In closing, it is worth noting that some work on this special problem was done previously by Toomre (Ref. 2) without using the slender body approximation. His numerical results were governed by a two-parameter family of solutions, the parameters being the friction coefficient and the average thickness ratio. As the analysis of this paper shows, the slender body approximation reduces the number of independent parameters to one, namely, the friction parameter K_f which combines the effects of the friction coefficient and thickness ratio.

5. LOGARITHMIC SPIRAL CONTOUR

It is now assumed that the transversal contour is a circle of radius r_o with n symmetric segments of a logarithmic spiral superimposed. In polar coordinates, the equation of each of these segments is given by

$$\frac{\dot{r}}{r} = \pm K \tag{36}$$

where K is a positive constant and where the dot sign denotes a derivative with respect to the argument θ. If r_m denotes the maximum radius and if the angular coordinate θ is measured from the r_m-direction, then Eq. (36) is valid in the interval

$$-\frac{\pi}{n} \leq \theta \leq \frac{\pi}{n} \tag{37}$$

with the understanding that the positive sign preceding K is to be employed for negative values of θ and the negative sign, for positive values of θ. Consequently, integration of Eq. (36) yields the following relationship between the parameter K, the number of segments n, and the ratio $y = r_m/r_o$ of maximum to minimum radius

$$K = \frac{n}{\pi} \log y \tag{38}$$

After the integrals (14) and (15) are evaluated, the following results can be readily obtained:

$$K_{Dp} = \frac{\pi r_o^4}{2(1 + K^2)} \frac{y^4 - 1}{\log y}$$

$$K_{Df} = C_f \pi r_o \sqrt{1 + K^2} \frac{y - 1}{\log y} \tag{39}$$

$$S = \frac{\pi r_o^2}{2} \frac{y^2 - 1}{\log y}$$

Consequently, the drag per unit square length and dynamic pressure becomes

$$K_D = \frac{2S^2}{\pi} \left(\frac{\alpha}{x} + 2K_f \beta \sqrt{x} \right) \tag{40}$$

where

$$\alpha = \frac{y^2 + 1}{y^2 - 1} \log y$$

$$\beta = \sqrt{\frac{2}{\log y} \frac{y - 1}{y + 1}} \tag{41}$$

In this equation, the friction factor K_f is defined by Eq. (20), and the auxiliary variable x is defined as

$$x = 1 + K^2 = 1 + \left(\frac{n \log y}{\pi} \right)^2 \tag{42}$$

and is, therefore, monotonically related to the number of segments composing the body in the interval of interest $2 \leq n \leq \infty$. This means that the variable x satisfies the inequality

$$\gamma \leq x \leq \infty \tag{43}$$

where

$$\gamma = \frac{\pi^2 + 4 \log^2 y}{\pi^2} \tag{44}$$

and where the lower bound corresponds to n = 2 and the upper bound, to $n = \infty$.

It is of interest to evaluate the ratio of the drag of this body to that of the equivalent body of revolution (subscript R). This ratio is given by

$$\frac{D}{D_R} = \frac{\alpha + 2K_f \beta x \sqrt{x}}{x(1 + 2K_f)} \tag{45}$$

and, for each given value of y, attains a minimum for the value of x defined by

$$x = \left(\frac{\alpha}{\beta K_f}\right)^{2/3} \tag{46}$$

Once more, the optimum number of segments

$$n = \frac{\pi}{\log y} \sqrt{\left(\frac{\alpha}{\beta K_f}\right)^{2/3} - 1} \tag{47}$$

is such that the friction drag is twice the pressure drag; the associated drag ratio is then

$$\frac{D}{D_R} = \frac{3(\sqrt{\alpha}\ \beta K_f)^{2/3}}{1 + 2K_f} \tag{48}$$

providing that

$$K_f \leq \frac{\alpha}{\beta\gamma\sqrt{\gamma}} \tag{49}$$

Should the friction parameter not satisfy inequality (49), then the optimum solution would be as follows:

$$K_f \geq \frac{\alpha}{\beta\gamma\sqrt{\gamma}} \quad \left\{ \begin{array}{l} x = \gamma \\ n = 2 \\ \dfrac{D}{D_R} = \dfrac{\alpha + 2K_f\beta\gamma\sqrt{\gamma}}{\gamma(1 + 2K_f)} \end{array} \right. \tag{50}$$

6. DISCUSSION AND CONCLUSIONS

The results of the previous analysis are summarized graphically in Figs. 2 through 5, with Figs. 2 and 3 pertaining to a pyramidal body and Figs. 4 and 5 pertaining to a body whose cross section is a circle with n symmetric arcs of a logarithmic spiral superimposed. These graphs show that (a) the optimum configuration depends strongly on the friction parameter, and (b) that, for practical values of K_f, the amount of drag reduction is much smaller than that predicted when friction is neglected.

With regard to specific details, Fig. 2 shows that the optimum transversal contour of a pyramidal body is a triangle for $K_f \lessapprox 0.47$ and a circle for $K_f > 1$. For intermediate values of the friction parameter, the optimum number of sides satisfies the inequality $3 < n < \infty$ and increases as the friction parameter increases. As long as $K_f < 1$, the drag of the optimum pyramidal body is smaller than that of the equivalent body of revolution (Fig. 3); the drag reduction depends on the friction parameter and is in the order of 20% for values of K_f around 1/5.

For a body whose cross section is a circle with n symmetric segments of a logarithmic spiral superimposed, the optimum number of segments depends not only on the friction parameter but also on the ratio of the maximum to the minimum radius of the cross section (Fig. 4). For this reason, the following considerations refer to one particular case, that in which $y = 2$. For $K_f \lessapprox 0.9$, the optimum number of segments satisfies the inequality $2 < n < \infty$. As the friction parameter increases, the optimum number of segments decreases and becomes $n = 2$ for $K_f \cong 0.9$. For any higher value of the friction parameter, the optimum number of segments remains equal to two.

It is obvious that the present solutions are physically realistic only for a discrete set of values of the friction parameter, that is, those values which yield integer values for n. For any other value of the friction parameter, the optimum noninteger number of sides must be replaced by one of the two nearest integer values, that which yields the lower amount of drag. Finally, since the present solutions have been obtained by direct methods, they necessarily exhibit a greater drag than that of the variational solution associated with the same problem. Therefore, it is of considerable engineering interest to extend this analysis by lifting the arbitrary limitations imposed here on the class of bodies being investigated. Since the order of magnitude of the friction drag of a slender body is the same as that of the pressure drag and since the present approach indicates that it is important to account for the friction drag in determining the optimum transversal contour, it follows that it is desirable that a variational approach be undertaken with due consideration for both components of the drag, that is, the pressure drag and the friction drag.

ACKNOWLEDGMENT

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APPENDIX

SIMULTANEOUS OPTIMIZATION OF THE LONGITUDINAL AND TRANSVERSAL CONTOURS

In the previous sections, the transversal contour of the body was optimized under the assumption that the longitudinal contour is conical. If this restriction is eliminated, one can optimize the longitudinal and the transversal contours simultaneously. As an example, consider a body whose longitudinal contour is described by the power law

$$f(z) = z^m \tag{51}$$

and whose transversal contour is a polygon. The evaluation of the drag integral (10) leads to

$$K_D = \frac{2S^2}{\pi}\left(\frac{m^3}{2m - 1}\ \frac{1}{x} + \frac{4K_f\sqrt{x}}{m + 1}\right) \tag{52}$$

which simplifies to

$$K_D = \frac{2S^2}{\pi}(1 + 2K_f) \tag{53}$$

for a circular cone. Consequently, the ratio of the drag of the body under consideration to that of the circular cone (subscript R) becomes

$$\frac{D}{D_R} = \frac{1}{1+2K_f}\left(\frac{m^3}{2m - 1}\,\frac{1}{x} + \frac{4K_f\sqrt{x}}{m + 1}\right) \tag{54}$$

where

$$1 \le x = \frac{n}{\pi} \tan \frac{\pi}{n} \le \frac{3\sqrt{3}}{\pi} \tag{55}$$

Clearly, then, the optimum combination of longitudinal and transversal contours is supplied by the relationships

$$\frac{\partial D}{\partial m} = 0 \quad , \quad \frac{\partial D}{\partial x} = 0 \tag{56}$$

which can be rewritten in the form

$$\begin{aligned} \frac{m^2(4m - 3)}{(2m - 1)^2} \frac{1}{x} - \frac{4K_f\sqrt{x}}{(m + 1)^2} &= 0 \\ \frac{m^3}{2m - 1} \frac{1}{x} - \frac{2K_f\sqrt{x}}{m + 1} &= 0 \end{aligned} \tag{57}$$

If these equations are regarded as linear and homogeneous in $1/x$ and $K_f\sqrt{x}$, it becomes apparent that the optimum longitudinal contour is described by

$$\begin{vmatrix} \frac{m^2(4m - 3)}{(2m - 1)^2} & \frac{4}{(m + 1)^2} \\ \frac{m^3}{2m - 1} & \frac{2}{m + 1} \end{vmatrix} = 0 \tag{58}$$

whose solution is the conical body

$$m = 1 \tag{59}$$

Since the optimum transversal contour satisfies the relationship

$$x = \frac{1}{K_f^{2/3}} \tag{60}$$

the optimum number of sides and the associated drag ratio are given by

$$K_f = \left[\frac{\pi}{n} \cot \frac{\pi}{n}\right]^{3/2}$$
$$\frac{D}{D_R} = \frac{3K_f^{2/3}}{1 + 2K_f} \tag{61}$$

provided that

$$\frac{\pi\sqrt{\pi}}{9\sqrt[4]{3}} \leq K_f \leq 1 \tag{62}$$

Should the friction parameter not satisfy the above inequality, then the transversal contour would either be a triangle or a circle while the optimum longitudinal contour would be described by the first of Eqs. (56) or (57). The simple manipulations, omitted for the sake of brevity, lead to the following results:

$$K_f \le \frac{\pi\sqrt{\pi}}{9\sqrt[4]{3}} \qquad \left\{ \begin{aligned} x &= \frac{3\sqrt{3}}{\pi} \\ n &= 3 \\ K_f &= \frac{\pi\sqrt{\pi}}{9\sqrt[4]{3}} \frac{4m-3}{4} \left[\frac{m(m+1)}{2m-1}\right]^2 \\ \frac{D}{D_R} &= \frac{1}{1+2K_f}\left[\frac{\pi}{3\sqrt{3}} \frac{m^3}{2m-1} + \frac{\sqrt[4]{27}}{\sqrt{\pi}} \frac{4K_f}{m+1}\right] \end{aligned} \right. \tag{63}$$

$$K_f \ge 1 \qquad \left\{ \begin{aligned} x &= 1 \\ n &= \infty \\ K_f &= \frac{4m-3}{4} \left[\frac{m(m+1)}{2m-1}\right]^2 \\ \frac{D}{D_R} &= \frac{1}{1+2K_f}\left[\frac{m^3}{2m-1} + \frac{4K_f}{m+1}\right] \end{aligned} \right. \tag{64}$$

Since the conclusions relative to the transversal contour are identical with those derived for conical bodies, the pertinent result is represented in Fig. 2. Concerning the longitudinal contour, the optimum exponent m is plotted in Fig. 6 versus the friction parameter. Furthermore, the minimum drag is represented in Fig. 7. From these graphs, it appears that the optimum longitudinal contour follows the 3/4-power law for $K_f = 0$, is convex for $0 < K_f \gtrsim 0.47$, is a straight line for $0.47 \gtrsim K_f < 1$, and is concave for $K_f > 1$. In conclusion, the effect of friction is to modify not only the optimum transversal contour but also the optimum longitudinal contour to a considerable degree.

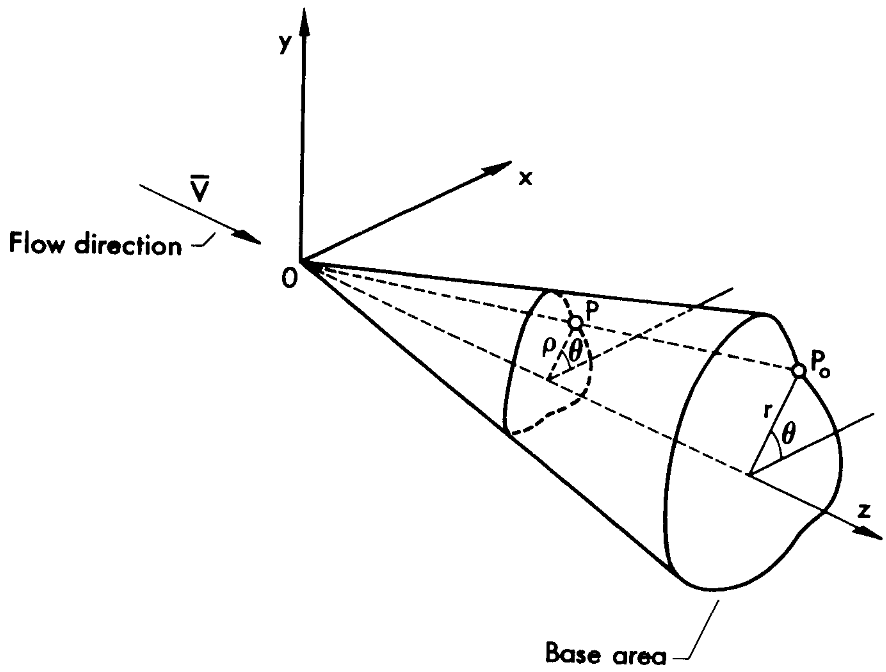

Fig. 1. Coordinate systems.

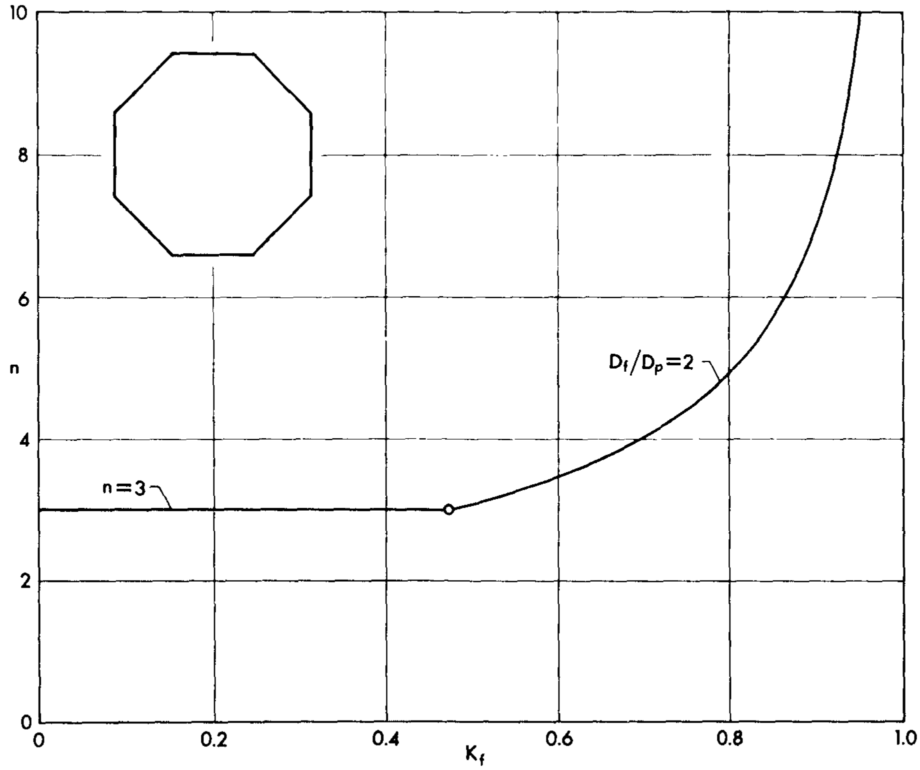

Fig. 2. Optimum number of sides of a pyramidal body.

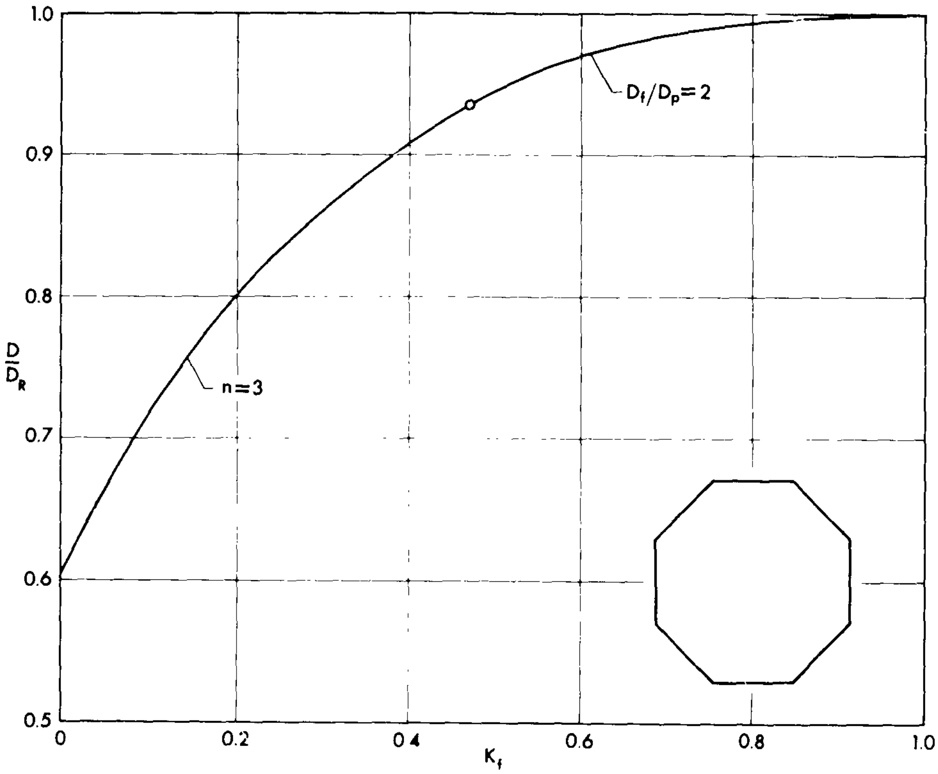

Fig. 3. Minimum drag of a pyramidal body.

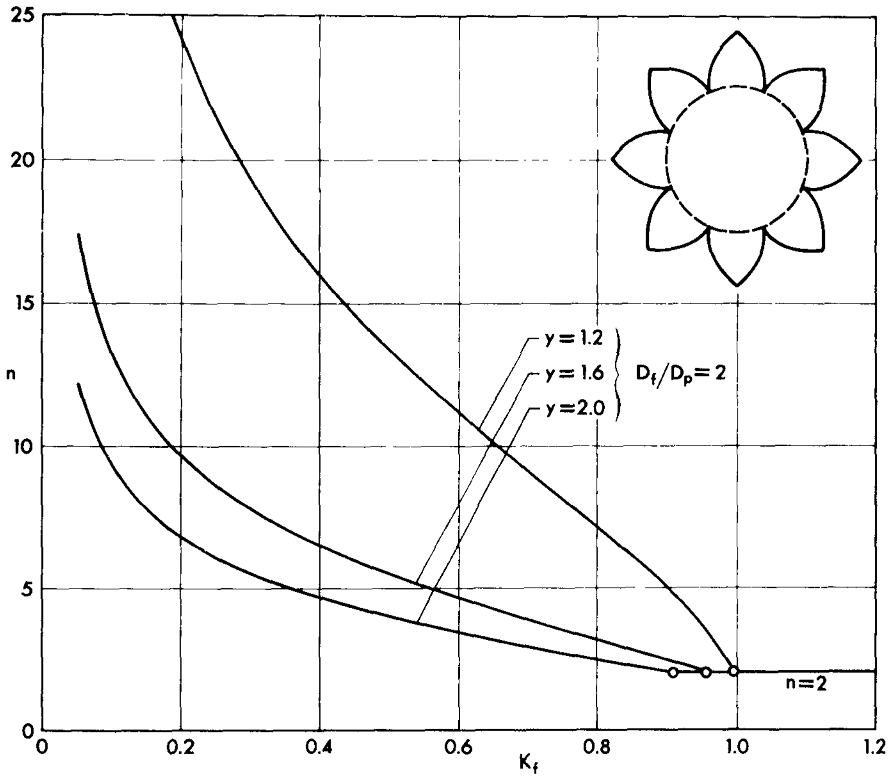

Fig. 4. Optimum number of segments of a "logarithmic spiral" body.

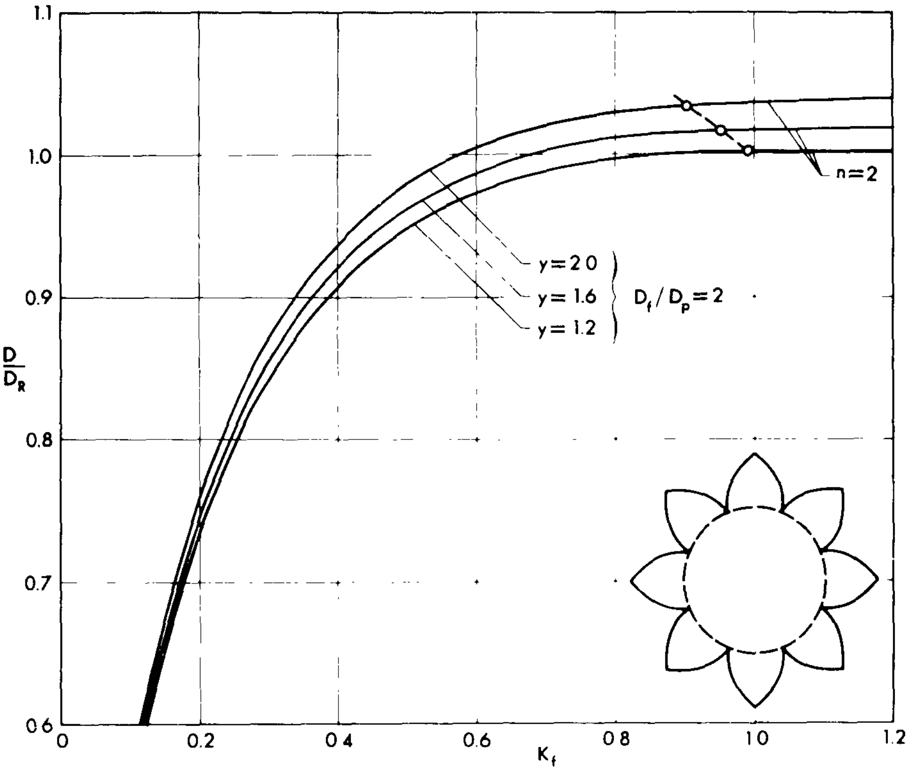

Fig. 5. Minimum drag of a "logarithmic spiral" body.

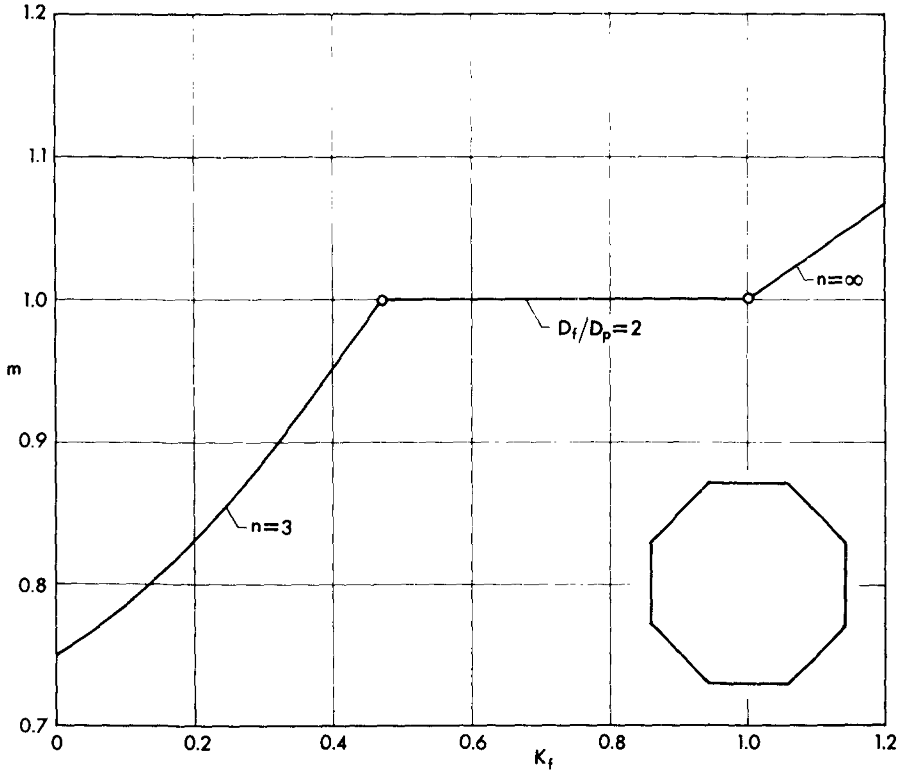

Fig. 6. Optimum exponent of a power body with a polygonal cross section.

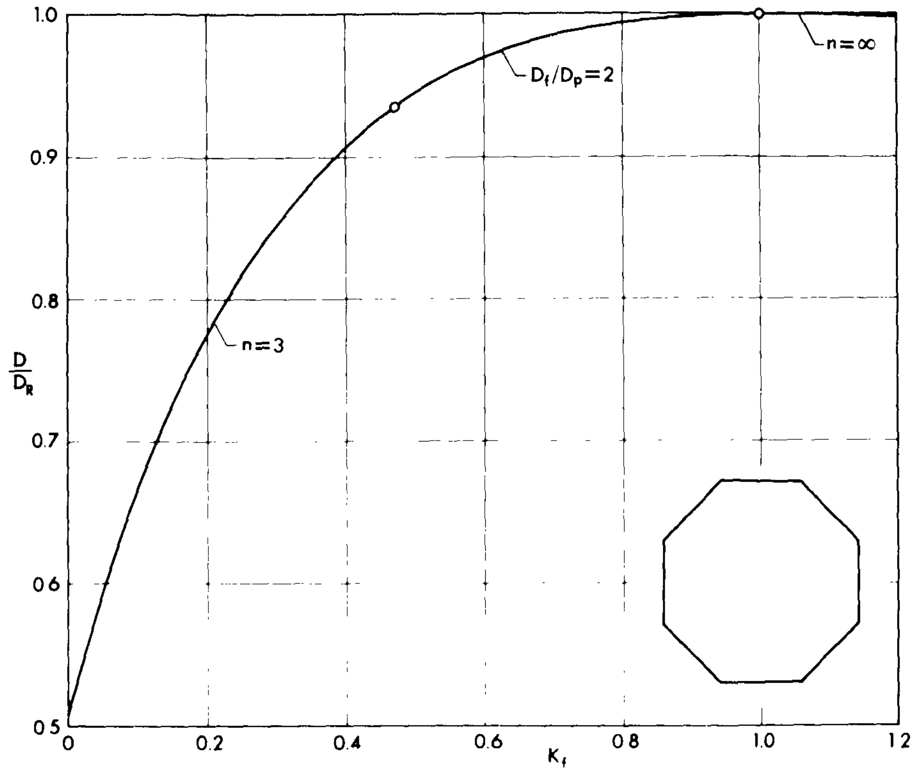

Fig. 7. Minimum drag of a power body with a polygonal cross section.